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Workshop of Graph Labelings

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Graph Labeling

- $G$ is a finite and simple graph.
  
  $V(G) =$ vertex set; $|V(G)| = p,$  
  $E(G) =$ edge set; $|E(G)| = q.$

- A labeling of a graph $G$ is a one to one mapping from some set of graph elements to a set of positive integers.
  
  - A vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, p\}.$
  - An edge labeling $f : E(G) \rightarrow \{1, 2, 3, \ldots, q\}.$
  - A total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}.$
Let $f$ be a total labeling of $G$.

- **Vertex-weight** $w(v), v \in V(G)$:
  
  Sum of label of $v$ and labels of its incident edges;
  \[
  w(v) = f(v) + \sum_{u \in N(v)} f(uv).
  \]

- **Edge-weight** $w(e), e = uv \in E(G)$:
  
  Sum of label of $e$ and of labels of its endpoints;
  \[
  w(uv) = f(u) + f(uv) + f(v).
  \]
Magic (Antimagic) Labeling

- A vertex-magic (vertex-antimagic) total labeling.
- An edge-magic (edge-antimagic) total labeling.
A vertex-magic total labeling of a graph $G$ with $p$ vertices and $q$ is a bijective function

$$f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$$

such that $w(x) = f(x) + \sum_{y \in N(x)} f(xy) = k_f$ is a constant for any vertex $x \in V(G)$.

$G$ is called a vertex-magic total graph.

$k_f$ is called magic constant of $f$.

Vertex-Antimagic Total Labeling

- An \((a, d)\)-vertex-antimagic total labeling of a graph \(G\) with \(p\) vertices and \(q\) is a bijective function

\[
f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p + q\}
\]

such that \(\{f(x) + \sum_{y \in N(x)} f(xy) : x \in V(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}\) for two integers \(a > 0\) and \(d \geq 0\).

- \(G\) is called an \((a, d)\)-vertex-antimagic total graph.

Edge-Antimagic Total Labeling

- An \((a, d)\)-edge-antimagic total labeling of a graph \(G\) with \(p\) vertices and \(q\) is a bijective function

\[
f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}
\]

such that \(\{f(x) + f(xy) + f(y) : xy \in E(G)\} = \{a, a + d, a + 2d, \ldots, a + (q - 1)d\}\) for two integers \(a > 0\) and \(d \geq 0\).

- \(G\) is called an \((a, d)\)-edge-antimagic total graph.

Figure: (a). A VMT graph with $k = 12$. (b). A (9, 2)-VAT graph. (c) A (9, 2)-EAT graph. (d) An EMT graph $k = 12$. 
Edge-Magic Total Labeling

- An edge-magic total labeling (EMTL) of a graph $G$ with $p$ vertices and $q$ edges is a one to one mapping

$$f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}$$

such that $f(x) + f(xy) + f(y) = k_f$ is a constant for any edge $xy$ of $G$.

- $G$ is called an edge-magic total graph.
- $k_f$ is called magic constant of $f$.

- **Conjecture**: Every tree is edge-magic total.

Super Edge-Magic Total Labeling

- An edge-magic total labeling $f$ of $G$ is called a super edge-magic total labeling (SEMTL) if $f(V(G)) = \{1, 2, 3, \ldots, p\}$.
- $G$ is called a super edge-magic total graph.

- Conjecture: Every tree is super edge-magic total.

Example

Figure: An edge-magic total graph with $k = 12$ and a super edge-magic total graph $k = 18$. 
Elementary Counting

Let $G$ be a graph with $V(G) = \{x_1, x_2, \ldots, x_p\}$ and $f$ be an EMTL of $G$.

$$\sum_{xy \in E(G)} [f(x) + f(xy) + f(y)] = kq$$

This sum contains each edge label once, and each vertex label $d_i$ times, where $d_i$ is degree of the vertex $x_i$. Thus,

$$kq = \frac{1}{2}(p + q)(p + q + 1) + \sum (d_i - 1)f(x_i) \ldots (*)$$

If $q$ is even, $p + q \equiv 2 \pmod{4}$, and every $d_i$ is odd, then $(*)$ is impossible.
Elementary Counting

- If $G$ has $q$ even, $p + q \equiv 2 \pmod{4}$, and every vertex has odd degree, then $G$ has no EMTL. [1]
  - The complete graph $K_n$ is not EMT when $n \equiv 4 \pmod{8}$.
  - The Wheel $W_n$ is not EMT when $n \equiv 3 \pmod{4}$.
  - The graph $tK_n$, consisting $t$ disjoin copies of $K_n$, is not EMT when $n \equiv 4 \pmod{8}$ and $t$ is odd.
  - The Wheel $tW_n$ is not EMT when $n \equiv 3 \pmod{4}$ and $t$ is odd.

Research Problem. [2]. Investigate graphs $G$ for which equation (*) implies the non-existence of an EMTL of $2G$.

Elementary Counting

Equation (*) may be used to provide bounds of magic constant $k$.

Let $d_1 \leq d_2 \leq \ldots \leq d_p$, then

$$kq \leq \left\lfloor \frac{1}{2} (p + q)(p + q + 1) + d_1(q + 1) + d_2(q + 2) + \ldots + d_p(p + q) \right\rfloor$$

and

$$kq \geq \left\lceil \frac{1}{2} (p + q)(p + q + 1) + d_1(p) + d_2(p + 1) + \ldots + d_p(1) \right\rceil.$$
Example: If $G = K_3$, then $k = 7 + \frac{1}{3} \sum_{i=1}^{3} f(x_i)$. So, $9 \leq k \leq 12$.

- $k = 9$, $\{f(x_1), f(x_2), f(x_3)\} = \{1, 2, 3\}$.
- $k = 10$, $\{f(x_1), f(x_2), f(x_3)\} = \{1, 3, 5\}$.
- $k = 11$, $\{f(x_1), f(x_2), f(x_3)\} = \{2, 4, 6\}$.
- $k = 12$, $\{f(x_1), f(x_2), f(x_3)\} = \{4, 5, 6\}$. 
Elementary Counting

**Example:** If $G = K_5$, then $k = 12 + \frac{3}{10} \sum_{i=1}^{5} f(x_i)$. So, $18 \leq k \leq 30$.

- $k = 18$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{1, 2, 3, 5, 9\}$.
- $k = 24$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{1, 8, 9, 10, 12\}$.
- $k = 24$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{4, 6, 7, 8, 15\}$.
- $k = 30$, $\{f(x_1), f(x_2), \ldots, f(x_5)\} = \{7, 11, 13, 14, 15\}$.
- $k = 21, 27$ no solutions.
Dual Labeling

- If $f$ is an EMTL of a graph $G$ with magic constant $k$, then $f'(u) = (p + q + 1) - f(u), u \in V(G) \cup E(G)$, is an EMTL of $G$ with magic constant $k' = 3(p + q + 1) - k$. [1]

- If $f$ is a SEMTL of a graph $G$ with magic constant $k$, then

$$f'(u) = \begin{cases} p + 1 - f(u), & \text{if } u \in v(g), \\ 2p + q + 1 - f(u), & \text{if } u \in E(G). \end{cases}$$

is a SEMTL of $G$ with magic constant $k' = 4p + q + 3 - k$. [2]


Some Known EMT Graphs

- Cycle $C_n$ for any $n \geq 3$.
- Wheel $W_n$ for any $n \equiv 0, 1, 2 \pmod{4}$.
- Fan $F_n$ for any $n \geq 2$.
- Sun $C_n \odot K_1$ for any $n \geq 3$.
- Complete bipartite graph $K_{n,m}$ for an $n \geq 1$ and $m \geq 1$.
- Book $B_{3,n}$, a graph consists of $n$ triangles with a common edge, for any $n \geq 1$.
- Generalize Petersen Graph $P(n, m)$ for odd $n \geq 3$.

Research Problems:

- The book $B_{m,n}$ consists of $n$ copies of $C_m$ with a common edge. Are all book $B_{m,n}$ EMT?
- An $(n, t)$-kite consists of a cycle $C_n$ with $t$-edge path attached to one vertex. Investigate the EMT properties of $(n, t)$-kites for any $t$. 
Some Known EMT Graphs

- If $G$ is a 3-colorable EMT graph, then $mG$ is EMT graph for every odd $m \geq 3$.

$3 \times m$ array Kotzig is

$$A = \begin{bmatrix}
0 & 1 & \ldots & s-1 & s & s+1 & \ldots & 2s-1 & 2s \\
s+1 & s+2 & \ldots & 2s & 0 & 1 & \ldots & s-1 & s \\
2s-1 & 2s-3 & \ldots & 1 & 2s & 2s-2 & \ldots & 2 & 0
\end{bmatrix}$$

$$B = \begin{bmatrix}
1 & 2 & \ldots & s & s+1 & s+2 & \ldots & 2s & 2s+1 \\
s+2 & s+3 & \ldots & 2s+1 & 1 & 2 & \ldots & s & s+1 \\
2s & 2s-2 & \ldots & 2 & 2s+1 & 2s-1 & \ldots & 3 & 1
\end{bmatrix}$$

Research Problems:
- Is $mW_n$ an EMT graph when $n \equiv 1 \pmod{4}$ and $m$ is odd?

(Super) Edge-Magic Total Strength

The \textit{(super) edge-magic total strength} of a graph \( G \), \( (s)\text{emt}(G) \), is the minimum of all magic constant \( k \) where the minimum is taken over all (S)EMTL of \( G \). That is,

\[
(s)\text{emt}(G) = \min\{k : f \text{ is an (S)EMTL of } G\}.
\]

Perfect (super) Edge-Magic Total Graph

- Let $T_G = \{ \frac{\sum_{u \in V(G)} d(u)f(u) + \sum_{e \in E(G)} f(e)}{q} \mid f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\} \text{ is a bijection} \}$.

- (Super) magic interval of $G$,
  $$J_G = \{ \lfloor \min T_G \rfloor, \lfloor \min T_G \rfloor + 1, \ldots, \lfloor \max T_G \rfloor \}.$$

- (Super) magic set of $G$,
  $$\sigma_G = \{ k \in J_G \mid k \text{ is magic constant of some (S)EMTL of } G \}.$$

- $G$ is perfect (S)EMT if $J_G = \sigma_G$.

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Super Edge-Magic Total Tree

Figure: Is this a tree?

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Super Edge-Magic Total Tree

- Path
- Star
- Caterpillar
- Tree with at most 17 vertices
- Path-like-tree
- Symmetric binary tree
- Firecracker
- Banana tree

Super Edge-Magic Total Tree

Figure: A path and path-like-trees?
More Results on Super Edge-Magic Total Labeling

- If \( G \) is SEMT, then \( q \leq 2p - 3 \). [1].
  - \( W_n \) is not SEMT for any \( n \).
  - \( K_n \) is not SEMT for any \( n \geq 3 \).
  - No regular graph of degree greater than 3 can be SEMT.
  - Every SEMT graph contains at least two vertices of degree less than 4.

- A graph \( G \) is SEMT if and only if there is a bijective function
  \[ f : V(G) \rightarrow \{1, 2, 3, \ldots , p\} \]
  such that the set \( S = \{f(x) + f(y) | xy \in E(G)\} \) is a set of \( q \) consecutive integers. [2].
  - If degree of every vertex of \( G \) is even and \( q \equiv 2 \pmod{4} \), then
    is not a SEMT graph.


More Results on Super Edge-Magic Total Labeling

- If a graph $G$ that is a tree or where $q \geq p$ is SEMT, then $G$ is sequential, harmonious, cordial. [1].
- Suppose that $G$ is a SEMT bipartite graph with partite sets $V_1$ and $V_2$ and let $f$ be a SEMTL of $G$ such that $f(V_1) = \{1, 2, 3, \ldots, |V_1|\}$, then $G$ has an $\alpha$-valuation. [1].


1. A sun $C_n \odot K_1$ is a graph constructed from a cycle $C_n$ by attaching one pendant to each vertex of the cycle $C_n$. Find the (super) edge-magic total strength of $C_3 \odot K_1$.
2. Prove that the sun $C_3 \odot K_1$ is a perfect (super) edge-magic total graph.
3. Find all possible values of magic constant for an edge-magic total labeling of $K_{1,m}$.
4. The graph $W_n - \{e\}$ is constructed from a wheel $W_n$ by deleting an edge. Which $W_n - \{e\}$ are super edge-magic total?
Results on 2-regular Graphs

Holden et al. proved that,

- $C_5 \cup (2t)C_3$ is SEMT for each integer $t \geq 3$.
- $C_4 \cup (2t - 1)C_3$ is SEMT for each integer $t \geq 3$.
- $C_7 \cup (2t)C_3$ is SEMT for each integer $t \geq 1$.

- **Conjectured**: The super edge-magic deficiency of all 2-regular graphs of odd order is **zero**, excluding $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$.

Results on 2-regular Graphs

Figueroa-Centeno, Ichishima, and Muntaner-Batle proved that:

- $C_3 \cup C_n$ is SEMT iff $n \geq 6$ and $n$ is even.
- $C_4 \cup C_n$ is SEMT iff $n \geq 5$ and $n$ is odd.
- $C_5 \cup C_n$ is SEMT iff $n \geq 4$ and $n$ is even.
- $C_n \cup C_m$ is SEMT if $n$ is even and $m \geq \frac{n}{2} + 1$ is odd.

Results on 2-regular Graphs

Let $G \cong \bigcup_{i=1}^{k} C_{n_i}$, $H \cong \bigcup_{i=1}^{k} (m, n_i)C_{[m,n_i]}$, and let $m$ be odd. If $G$ is SEMT, then $H$ is SEMT.

- $(a, b)$ is the greatest common divisor of $a$ and $b$,
- $[a, b]$ is the least common multiple of $a$ and $b$.

Results on 2-regular Graphs

A graph \( G \cup tK_1 \) is called \textit{pseudo super edge-magic total} (PSEMT) if there exists a bijection \( f : V(G \cup kK_1) \to \{1, 2, 3, \ldots |V(G)| + t\} \) such that the set \( \{f(x) + f(y) : xy \in E(G)\} \cup \{2f(u) : u \in kK_1\} \) is a set of \( |E(G)| + t \) consecutive integers. In such a case \( f \) is called a \textit{pseudo super edge-magic total labeling} (PSEMTL) of \( G \cup kK_1 \).

- Let \( G \cong \bigcup_{i=1}^{k} C_{n_i} \cup tK_1 \), \( H \cong \bigcup_{i=1}^{k} (m, n_i)C_{[m,n_i]} \cup tC_m \), and let \( m \) be odd. If \( G \) is PSEMT, then \( H \) is SEMT.
  - \((a, b)\) is the \textit{greatest common divisor} of \( a \) and \( b \),
  - \([a, b]\) is the \textit{least common multiple} of \( a \) and \( b \).

Results on 2-regular Graphs

Cichacz et al. [1] (2017) introduced a technique for constructing vertex-magic total labelings of 2-regular graphs and proved the following results, which contributes significantly to Holden et al. [2] conjecture.

- [1] If \( G \cong \bigcup_{i=1}^{k} C_{n_i} \) is SEMT, then \( H \cong \bigcup_{i=1}^{k} C_{mn_i} \) is SEMT for every odd \( m \).


In 1970, Kotzig and Rosa proved that for every graph $G$ there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$.

- The edge-magic deficiency of a graph $G$, $\mu(G)$, is defined as $\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic}\}$.

- $\mu(G) \leq F_{p+2} - 2 - p - \frac{1}{2}p(p - 1)$, where $p = |V(G)|$ and $F_p$ is the $p$th Fibonacci number.

Super Edge-Magic Deficiency

- Let $G$ be a graph and let

$$M(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is super edge-magic}\}.$$ 

The super edge-magic deficiency of a graph $G$, $\mu_s(G)$, is defined to be

$$\mu_s(G) = \begin{cases} 
\min M(G), & \text{if } M(G) \neq \emptyset, \\
+\infty, & \text{if } M(G) = \emptyset.
\end{cases}$$

- $\mu_s(G)$ measure how “close” a graph to be a super edge-magic graph.
- $\mu(G) \leq \mu_s(G)$, for every graph $G$.

Example

Figure: (a) $\mu(C_4) = 0$, (b) $\mu_s(C_4) = 1$, (c) $\mu(K_4) = \mu_s(K_4) = 1$
Graph $G$ with $\mu_s(G) = +\infty$

Let $G$ be a graph of size $q$ such that $\text{deg}(v)$ is even for all $v \in V(G)$ and $q \equiv 2 \pmod{4}$, then $\mu_s(G) = +\infty$.

$\mu_s(C_n) = +\infty$, if $n \equiv 2 \pmod{4}$.

Graph $G$ with $\mu_s(G) = \infty$

A set $X = \{x_1 < x_2 < \cdots < x_n\} \subseteq \mathbb{N}$ is a well-spread set (WS-set for short) if the sums $x_i + x_j$ for $i < j$ are all different.

The smallest span of pairwise sums of cardinality $n$, denoted by $\rho^*(n)$,

$\rho^*(n) = \min\{x_n + x_{n-1} - x_2 - x_1 : \{x_1 < x_2 < \cdots < x_n\}\text{ is a WS-set}\}$.

- The smallest span of pairwise sums of cardinality $n$, $\rho^*(n)$ satisfies: $\rho^*(4) = 6$, $\rho^*(5) = 11$, $\rho^*(6) = 19$, $\rho^*(7) = 30$, $\rho^*(8) = 43$ and $\rho^*(n) > n^2 - 5n + 14$ for $n > 9$.

Graph $G$ with $\mu_s(G) = +\infty$

- Let $G$ be a graph that contains the complete subgraph $K_n$. If $|E(G)| < \rho^*(n)$, then $\mu_s(G) = +\infty$.
- For any positive integer $n$,

$$\mu_s(K_n) = \begin{cases} 
0, & \text{if } n = 1, 2, 3, \\
1, & \text{if } n = 4, \\
+\infty, & \text{for otherwise.}
\end{cases}$$

Super Edge-Magic Deficiency of Forests

- Let $F$ be a forest, then $\mu_s(F) \leq +\infty$.
- For every positive integer $n$,
  \[ \mu_s(nK_2) = \begin{cases} 
  0, & \text{if } n \text{ is odd}, \\
  1, & \text{if } n \text{ is even}. 
\end{cases} \]

Super Edge-Magic Deficiency of Forests

- For every two positive integers \( m \) and \( n \),

\[
\mu_s(P_m \cup K_{1,n}) = \begin{cases} 
1, & \text{if } m = 2 \text{ and } n \text{ is odd,} \\
& \text{or } m = 3 \text{ and } n \not\equiv 3 \pmod{3}, \\
0, & \text{otherwise.}
\end{cases}
\]

- For every two positive integers \( m \) and \( n \),

\[
\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 
0, & \text{either } m \text{ is a multiple of } n + 1, \\
& \text{or } n \text{ is a multiple of } m + 1, \\
1, & \text{otherwise.}
\end{cases}
\]

Super Edge-Magic Deficiency of Forests

- For every two positive integers $m$ and $n$,

$$\mu_s(P_m \cup P_n) = \begin{cases} 
1, & \text{if } (m, n) \in \{(2, 2), (3, 3)\}, \\
0, & \text{otherwise.}
\end{cases}$$

- **Conjecture**: If $F$ is a forest with two components then $\mu_s(F) \leq 1$.

Super Edge-Magic Deficiency of Forests

A banana tree $BT(n_1, n_2, \ldots, n_k)$ is a tree obtained from the stars $K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_k}$ by joining a new vertex to a single leave of each star.

- Let $G \cong BT(n_1, n_2) \cup BT(m_1, m_2, \ldots, m_k)$. For $n_2 = 2k$, $n_1 \geq n_2 \geq m_1 \geq m_2 \geq \ldots \geq m_k$ and $|V(G)| \equiv 2 \pmod{4}$, then $\mu_s(G) \leq 1$.
- For every $n \geq 3$, $\mu_s(K_{1,n-1} \cup K_{1,n} \cup K_{1,5n-12}) = 0$.
- For $k \geq 3$ is odd, $n \geq 5$, and $m \geq 3$, $\mu_s(P_n \cup P_{n+k} \cup K_{1,m}) \leq n + \lfloor \frac{k}{2} \rfloor$.
- For $n \geq 4$ and $m \geq 3$, $\mu_s(P_{n-1} \cup P_n \cup K_{1,m}) \leq n - 1$.
- For $n \geq 3$ and $m \geq 3$, $\mu_s(P_n \cup P_{2n} \cup K_{1,m}) \leq \lfloor \frac{3n}{2} \rfloor - 1$.

Super Edge-Magic Deficiency of Forests

Imran and Mukhtar (2017) proved that $\mu_s(G) = 0$ if

- $G \cong T(n, n, n, n + 1) \cup K_{1, \frac{1}{2}(n-2)}, \ 3 \leq n \equiv 1 \ (\text{mod} \ 2),$
- $G \cong T(n, n - 1, t, t + 2) \cup K_{1, \frac{t}{2}}, \ t \geq n$ and $n, t \equiv 0 \ (\text{mod} \ 2),$
- $G \cong T(n, n - 1, t, t + 2, 2t + 4) \cup K_{1, t+1}, \ t \geq n \geq 4$ and $n, t \equiv 0 \ (\text{mod} \ 2),$
- $G \cong T(n, n - 1, t, t + 2, 2t + 4, 4t + 8) \cup K_{1, 2t+3}, \ t \geq n \geq 2$ and $n, l \equiv 0 \ (\text{mod} \ 2),$
- $G \cong T(n, n - 1, t, t + 2, \ldots, t_r) \cup K_{1, \frac{t}{2}}, \ t \geq n \geq 2, \ n, t \equiv 0 \ (\text{mod} \ 2),$ and $4 \leq t_r = 2^{r-4}(t + 2)r,$

where $T(n_1, n_2, \ldots, n_r)$ is a graph obtained by replacing each edge of a star $K_{1,n}$ by a path of length $n_1, n_2, \ldots, n_r,$ respectively.

Super Edge-Magic Deficiency of $K_{n,m}$

- For every two positive integers $m$ and $n$,\[ \mu_s(K_{n,m}) \leq (n - 1)(m - 1). \]
- For any positive integer $m$, $\mu_s(K_{2,m}) = m - 1$ .

- **Conjecture**: For every two positive integers $m$ and $n$, \[ \mu_s(K_{n,m}) = (n - 1)(m - 1) \]

Super Edge-Magic Deficiency of $K_{n,m}$

- For $n = 2, 3$ and $4$ and for any positive integers $m$,
  \[ \mu_s(K_{n,m}) = (n - 1)(m - 1) \]  
  [1].

- For every two positive integers $m$ and $n$,
  \[ \mu_s(K_{n,m}) = (n - 1)(m - 1) \]  
  [2].

Super Edge-Magic Deficiency of $tK_{1,m}$

Figueroa-Centeno et al. (2005) [1]:
- For all positive integers $t$ and $n$ such that $t$ is odd,
  \[ \mu_s(tK_{1,n}) = 1. \]

Baskoro and AAGN (2003) [2]:
- For all even integers $t \geq 4$, $\mu_s(tK_{1,2}) = 0$.


Super Edge-Magic Deficiency of $tK_{1,m}$

- For all positive integers $t$ and $m$ such that $t$ is even, $\mu_s(tK_{1,m}) \leq 1$.
- For every positive integer $m$, $\mu_s(2K_{1,m}) = 1$.
- For all positive integers $t$ and $m$ such that $t \equiv 2 \pmod{4}$ and $m$ is odd, $\mu_s(tK_{1,m}) = 1$.
- For every positive integer $t$,

$$
\mu_s(tK_{1,3}) = \begin{cases} 
0, & \text{if } t \equiv 4 \pmod{8} \text{ or } t \text{ is odd}, \\
1, & \text{if } t \equiv 2 \pmod{4}.
\end{cases}
$$

- **Open Problem:** For even $t \geq 4$ and $m \geq 3$, determine the exact value of $\mu_s(tK_{1,m})$.

Super Edge-Magic Deficiency of $tK_{n,m}$

Simanjuntak, Baskoro, Uttunggadewa and AAGN (2008)[1]:

- For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 4$ and $m \geq 4$,
  $$\mu_s(tK_{n,m}) \leq t(nm - n - m) + 1.$$ 

Ichishima and Oshima (2011)[2]:

- For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 2$ and $m \geq 2$,
  $$\mu_s(tK_{n,m}) \leq t(nm - n - m) + 1.$$ 

- **Conjecture**: For all integers $t$, $n$ and $m$, with $t \geq 1$, $n \geq 2$ and $m \geq 2$,
  $$\mu_s(tK_{n,m}) = t(nm - n - m) + 1.$$ 


Super Edge-Magic Deficiency of $Q_n$

- For $n \geq 4$, $(n - 4)2^{n-2} + 3 \leq \mu_s(Q_n) \leq (n - 2)2^{n-1} - 4$.
- **Open Problem.** Determine the exact value of $\mu_s(Q_n)$.

For every integer $n \geq 3$,

$$\mu_s(C_n) = \begin{cases} 
0, & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
1, & \text{if } n \equiv 0 \pmod{4}, \\
+\infty, & \text{if } n \equiv 2 \pmod{4}.
\end{cases}$$

Super Edge-Magic Deficiency of 2-regular Graphs

- For every integer $n \geq 3$,

$$\mu_s(2C_n) = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
\infty, & \text{if } n \text{ is odd}.
\end{cases}$$

- For every integer $n \geq 3$,

$$\mu_s(3C_n) = \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
1, & \text{if } n \equiv 0 \pmod{4}, \\
\infty, & \text{if } n \equiv 2 \pmod{4}.
\end{cases}$$

- For $n \equiv 0 \pmod{4}$, $\mu_s(4C_n) = 1$.

Super Edge-Magic Deficiency of 2-regular Graphs

- **Conjecture:** For every integers \( m \geq 1 \) and \( n \geq 3 \),

\[
\mu_s(mC_n) = \begin{cases} 
0, & \text{if } mn \text{ is odd}, \\
1, & \text{if } mn \equiv 0 \pmod{4}, \\
+\infty, & \text{if } mn \equiv 2 \pmod{4}.
\end{cases}
\]

Super Edge-Magic Deficiency of 2-regular Graphs

Holden et al. proved that, for some integer $t$, $C_5 \cup (2t)C_3$, $C_4 \cup (2t-1)C_3$, and $C_7 \cup (2t)C_3$, are strong vertex-magic, which is in fact equivalent to saying that they are super edge-magic. In other words, they proved that

- $\mu_s(C_5 \cup (2t)C_3) = 0$ for each integer $t \geq 3$.
- $\mu_s(C_4 \cup (2t-1)C_3) = 0$ for each integer $t \geq 3$.
- $\mu_s(C_7 \cup (2t)C_3) = 0$ for each integer $t \geq 1$.

- **Conjectured**: The super edge-magic deficiency of all 2-regular graphs of odd order is zero, excluding $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$.

Super Edge-Magic Deficiency of 2-regular Graphs

Figueroa-Centeno, Ichishima, and Muntaner-Batle [1] proved that:

1. \( \mu_s(C_3 \cup C_n) = 0 \) iff \( n \geq 6 \) and \( n \) is even.
2. \( \mu_s(C_4 \cup C_n) = 0 \) iff \( n \geq 5 \) and \( n \) is odd.
3. \( \mu_s(C_5 \cup C_n) = 0 \) iff \( n \geq 4 \) and \( n \) is even.
4. \( \mu_s(C_n \cup C_m) = 0 \) if \( n \) is even and \( m \geq \frac{n}{2} + 1 \) is odd.

Ichishima and Oshima [2] investigated the super edge-magic deficiency of 2-regular graphs \( C_m \cup C_n \) for \( m = 3, 4, 5, 7 \) and any \( n \).


Super Edge-Magic Deficiency of 2-regular Graphs

- Let $G \cong \bigcup_{i=1}^{k} C_{n_i}$, $H \cong \bigcup_{i=1}^{k} (m, n_i)C_{[m,n_i]}$, and let $m$ be odd. If $\mu_s(G) = 0$, then $\mu_s(H) = 0$.

- $(a, b)$ is the greatest common divisor of $a$ and $b$,
- $[a, b]$ is the least common multiple of $a$ and $b$.

Super Edge-Magic Deficiency of 2-regular Graphs

Cichacz et al. [1] (2017) introduced a technique for constructing vertex-magic total labelings of 2-regular graphs and proved the following results, which contributes significantly to Holden et al. [2] conjecture.

- [1] If $\mu_s(\bigcup_{i=1}^{k} C_{n_i}) = 0$, then $\mu_s(\bigcup_{i=1}^{k} C_{m_{n_i}}) = 0$ for every odd $m$.


Super Edge-Magic Deficiency of Join Product Graphs

Join product of two vertex disjoin graphs $G$ and $H$, $G + H$, is their graph union with additional edges that connect all vertices of $G$ to each vertex of $H$.

- For any integers $n, m \geq 3$,
  $$\left[ \frac{1}{2}(n - 2)(m - 1) \right] \leq \mu_s(P_n + mK_1) \leq (n - 1)(m - 1) - 1.$$  

- For any integers $n, m \geq 2$,
  $$\left[ \frac{1}{2}(n - 1)(m - 1) \right] \leq \mu_s(K_{1,n} + mK_1) \leq n(m - 1) - 1.$$  

- For any integers $n \geq 3$ and $m \geq 2$,
  $$\mu_s(C_n + mK_1) \geq \left\lceil \frac{1}{2}(m + 1)n \right\rceil - (n + m) + 2.$$  

- For any integer $m \geq 2$ and odd integer $n \geq 3$,
  $$\mu_s(C_n + mK_1) \leq mn - (n + m) + 1.$$  

Super Edge-Magic Deficiency of Join Product Graphs

Let $G$ be a super edge-magic graph of order $p$ and size $q \geq 1$ with a super edge-magic labeling $f$. For any integer $m \geq 1$,

$$
\mu_s(G + mK_1) \leq \begin{cases} 
p + 1 - \min(S), & \text{if } m = 1, \\
(m - 2)(p - 1) + (q - 1), & \text{if } m \geq 2,
\end{cases}
$$

where $S = \{f(x) + f(y) : xy \in E(G)\}$.

**Open Problem:** Find a better upper bound of $\mu_s(G + mK_1)$ when $G$ is a super edge-magic graph.

Super Edge-Magic Deficiency of Join Product Graphs

1. Let $G$ be a graph with no cycle and isolated vertices. If $\mu_s(G + K_1) = 0$, then $G$ is a tree or a forest.
   - $\mu_s([K_{1,n} \cup K_2] + K_1) = 0$ iff $n = 2$. [1]
   - $\mu_s([P_n \cup K_2] + K_1) = 0$ iff $n = 3, 4, 5$. [1]
   - $\mu_s(DS_n + K_1) = 0, n \geq 1$, where $DS_n$ is a double star. [1]
   - $\mu_s(F_n = P_n + K_1) = 0$ iff $1 \leq n \leq 6$. [2]
   - $\mu_s(C^n_3 = nK_2 + K_1) = 0$ iff $n = 3, 4, 5, 7$. [3]


Super Edge-Magic Deficiency of Join Product Graphs

- Let $G$ be a tree of order $n \geq 7$ and $H \cong G + K_1$. If $\mu_s(H) = 0$ then either $K_3 \cup K_{1,3}$ or $2K_{1,3}$ is a subgraph of $H$.
- If $G$ is any tree of order $|V(G)| \leq 6$ excluding $G_1$ then $\mu_s(G + K_1) = 0$.
- $\mu_s(G_1 + K_1) = 1$.

Super Edge-Magic Deficiency of Join Product Graphs

- [1] Let $G$ be a tree and $m \geq 2$ be an integer.
  \[ \mu_s(G + mK_1) = 0 \text{ iff } G = P_2. \]

- [1] Let $G$ be a tree of order $n \geq 3$. For every positive integer $m \geq 2$,
  \[ \mu_s(G + mK_1) \geq \left\lfloor \frac{(m-1)(n-2) + 1}{2} \right\rfloor. \]

  - $\mu_s(P_4 + mK_1) = m - 1$ and $\mu_s(P_6 + mK_1) = 2(m - 1)$. [2]
  - $\mu_s(P_n + 2K_1) = \frac{n-2}{2}$ for any even integer $n \geq 2$. [3]

Let $G$ be a super edge-magic graph of order $p$ and size $q \geq 1$ with a super edge-magic labeling $f$. For any integer $m \geq 1$,

$$
\mu_s(G + mK_1) \leq \begin{cases} 
  p + 1 - \min(S), & \text{if } m = 1, \\
  (m - 2)(p - 1) + (q - 1), & \text{if } m \geq 2,
\end{cases}
$$

where $S = \{f(x) + f(y) : xy \in E(G)\}$.

**Open Problem:** Find a better upper bound of $\mu_s(G + mK_1)$ when $G$ is a super edge-magic graph.

Super Edge-Magic Deficiency of Join Chain Graphs

- A chain graph is a graph with blocks $B_1, B_2, B_3, \ldots, B_k$ such that for every $i$, $B_i$ and $B_{i+1}$ have a common vertex in such a way that the block-cut-vertex graph is a path.

- A chain graph with blocks $B_1, B_2, B_3, \ldots, B_k$ is denoted by $C[B_1, B_2, \ldots, B_k]$.

- If $B_1 = B_2 = \ldots = B_t \cong B$ then $C[B_1, B_2, \ldots, B_k]$ is denoted by $C[B^{(t)}, B_{t+1}, \ldots, B_k]$.

- If for every $i$, $B_i \cong H$ then $C[B_1, B_2, \ldots, B_k]$ is denoted by $kH$-path.

- Let $c_1, c_2, \ldots, c_{k-1}$ be the consecutive cut vertices of $C[B_1, B_2, \ldots, B_k]$. The string of $C[B_1, B_2, \ldots, B_k]$ is $(k-2)$-tuple $(d_1, d_2, \ldots, d_{k-2})$ where $d_i$ is the distance between $c_i$ dan $c_{i+1}$, $1 \leq i \leq k-2$. 
Super Edge-Magic Deficiency of Chain Graphs

(a) The chain graph $C[K_4^{(2)}, C_4^{(2)}, K_3^{(2)}]$ with string $(1,1,1,1)$.
(b) The chain graph $6C_4$-path with string $(2,1,2,1)$.
Super Edge-Magic Deficiency of Chain Graphs

- If $G \cong kK_3$-path then, for any integer $k \geq 3$,
  \[ \mu_s(G) = \begin{cases} 
  0, & \text{if } k \equiv 0, 1 \pmod{4}, \\
  +\infty, & \text{if } k \equiv 2 \pmod{4}, 
  \end{cases} \]
  and $\mu_s(G) \leq k - 1$, if $k \equiv 3 \pmod{4}$.

- If $G \cong kK_4$-path then, for any integer $k \geq 3$, $\mu_s(G) = 1$.

Super Edge-Magic Deficiency of Chain Graphs

- A diagonal ladder, $DL_m$, is a graph obtained from the ladder $L_m \cong P_m \times P_2$ by adding two diagonals in each rectangle of $L_m$.
- $\mu_s(DL_m) = \lceil \frac{m}{2} \rceil$, for every $m \geq 2$. [1]
- Let $H \cong kDL_m$-path with string $(d_1, d_2, \ldots, d_{k-2})$, where $d_i \in \{1, 2, \ldots, m - 1\}$.

![Diagram](image)

Figure: A $4DL_4$-path with string $(3, 3)$.

Super Edge-Magic Deficiency of Chain Graphs

- For any integers \( k \geq 3 \) and \( m \geq 2 \), \( \mu_s(H) \geq \left\lfloor \frac{1}{2}(m - 2)k + 1 \right\rfloor \).
- For any integers \( k \geq 3 \) and \( m \geq 2 \), if \( H \) has string \((m - 1, \ldots, m - 1)\) then \( \mu_s(H) = \left\lfloor \frac{1}{2}(m - 2)k + 1 \right\rfloor \).

Super Edge-Magic Deficiency of Chain Graphs

- Let $F \cong C[K_4^{(k)}, DL_m, K_4^{(n)}]$ with string $(1^{(k-1)}, d, 1^{(n-1)})$, where $d \in \{1, 2, 3, \ldots, m - 1\}$.

  - For any integers $k, n \geq 1$ and $m \geq 2$, $\mu_s(F) \geq \lceil \frac{m}{2} \rceil$.
  - For any integers $k, n \geq 1$ and $m \geq 2$, if $F$ has string $(1^{(k-1)}, m - 1, 1^{(n-1)})$ then $\mu_s(F) = \lfloor \frac{m}{2} \rfloor$.

Super Edge-Magic Deficiency of Chain Graphs

- A triangle ladder, $TL_m$, is a graph obtained from the ladder $L_m \cong P_m \times P_2$ by adding a single diagonal in each rectangle of $L_m$.
- $\mu_s(TL_m) = 0$, for every $m \geq 2$. [1]
- For $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = TL_m$ when $j$ is odd and $B_j = DL_m$ when $j$ is even.

![Diagram of a chain graph]

Figure: A chain graph $C[TL_4, DL_4, TL_4, DL_4]$ with string $(3, 4)$.

Super Edge-Magic Deficiency of Chain Graphs

- For any integer \( m \geq 3 \),

\[
\mu_s(G) \geq \begin{cases} 
\left\lfloor \frac{1}{4}k(m-3) \right\rfloor + 1, & \text{if } k \text{ is even,} \\
\left\lfloor \frac{1}{4}(k(m-3) - (m-1)) \right\rfloor + 1, & \text{if } k \text{ is odd.}
\end{cases}
\]

- If \( G \) has string \((m-1, d_1, m-1, d_2, m-1, \ldots, d_{\frac{1}{2}(k-3)}, m-1)\) when \( k \) is odd or \((m-1, d_1, m-1, d_2, \ldots, m-1, d_{\frac{1}{2}(k-2)})\) when \( k \) is even, where \( d_1, d_2, \ldots, d_{\left\lfloor \frac{1}{2}(k-2) \right\rfloor} \in \{m-1, m\} \),

then for any odd integer \( m \geq 3 \),

\[
\mu_s(G) = \begin{cases} 
\frac{1}{4}k(m-3) + 1, & \text{if } k \text{ is even,} \\
\frac{1}{4}(k-1)(m-3), & \text{if } k \text{ is odd.}
\end{cases}
\]

Super Edge-Magic Deficiency of Chain Graphs

Let $H \cong C[K_4^{(p)}, TL_m, K_4^{(q)}]$ with string $(1^{(p-1)}, d, 1^{(q-1)})$, where $d \in \{m - 1, m\}$. For any integers $p, q \geq 1$ and $m \geq 2$, $\mu_s(H) = 0$.

Super Edge-Magic Deficiency of Chain Graphs

- For $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = TL_n$ when $j$ is odd and $B_j = DL_m$ when $j$ is even, where $n$ is not necessarily equal to $m$.

  - If $G$ has string $(m - 1, d_1, m - 1, d_2, m - 1 \ldots, d_{\frac{1}{2}(k-3)}, m - 1)$ when $k$ is odd or $(m - 1, d_1, m - 1, d_2, \ldots, m - 1, d_{\frac{1}{2}(k-2)})$ when $k$ is even, where $d_1, d_2, \ldots, d_{\frac{1}{2}(k-2)} \in \{n - 1, n\}$, then for any integers $n \geq 2$ and $m \geq 3$ such that $m$ is odd,

    \[
    \mu_s(G) = \begin{cases} 
    \frac{1}{4}k(m - 3) + 1, & \text{if } k \text{ is even}, \\
    \frac{1}{4}(k - 1)(m - 3), & \text{if } k \text{ is odd}.
    \end{cases}
    \]

Super Edge-Magic Deficiency of Chain Graphs

- For every integer $k \geq 3$, let $G = C[B_1, B_2, \ldots, B_k]$, where $B_j = DL_m$ when $j$ is odd and $B_j = TL_n$ when $j$ is even.

  - If $G$ has string $(d_1, m - 1, d_2, m - 1, \ldots, m - 1, d_{\frac{1}{2}(k-2)})$, where $d_1, d_2, \ldots, d_{\frac{1}{2}(k-2)} \in \{n - 1, n\}$, then for any integers $n \geq 2$, $k \geq 3$ and $m \geq 3$ such that $k$ and $m$ are odd,

  $$\frac{1}{4}(k + 1)(m + 1) - k \leq \mu_s(G) \leq \frac{1}{4}(k + 1)(m + 1) - (k - 1).$$

References


THANK YOU FOR YOUR ATTENTION